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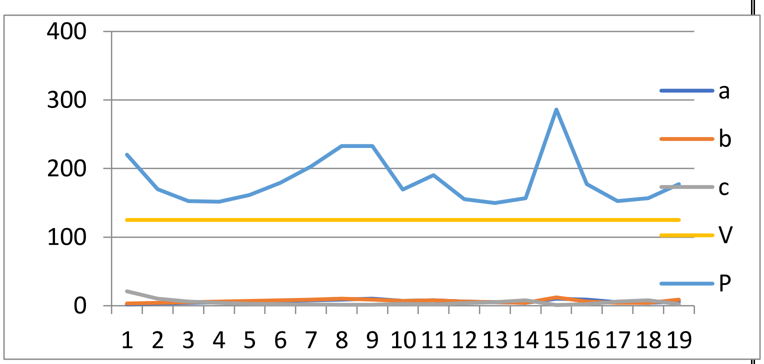
TOPIC PLAN		
Partner organization	Goce Delcev University – Stip, North Macedonia	
Topic	Function with Two Variables: Application of Derivatives	
Lesson title	Minimizing and Maximizing Problems	
Learning objectives	✓ Students will acquire and deal with derivatives of functions with several variables; ✓ Students will be able to estimate minimum and maximum of different sizes using differentiation of functions with two variables; ✓ Students will be able to deal with different problems in everyday life, which require calculating minimum or maximum value of a given size; ✓ Students are encouraged to use technology and different software in their work, while considering problem - based situations.	**Strategies/Activities** ☐Graphic Organizer ☐Think/Pair/Share ☐Modeling ☐Collaborative learning ☐Discussion questions ☐Project based learning ☐Problem based learning **Assessment for learning** ☐Observations ☐Conversations ☐Work sample ☐Conference ☐Check list ☐Diagnostics **Assessment as learning** ☐Self-assessment ☐Peer-assessment ☐Presentation ☐Graphic Organizer
Aim of the lecture / Description of the practical problem	The aim of the lecture is to make students able to calculate derivatives of a function with several variables and apply the derivatives to calculate minimum and maximum of given size. The teacher gives the next problems to the students: 1. *We have to make a tin tank in a form of rectangular cuboid that will collect 125 liters liquid. Which dimensions of the tank will require the least amount of material for its construction?* 2. *We have to make box which requires 200 cm^2 cardboard for its construction. Which dimensions should the box be in order its volume to be the largest possible?* The teacher divides students into two groups and associates a problem to each group. Students have to work and collaborate in order to find a solution of the problem.	

Previous knowledge assumed:	- derivatives of functions with one real variable - area and volume of geometric 3D-forms - algebraic equations - differentiation of functions of two real variables - determinants of second and third order	☐Homework **Assessment of learning** ☐Test ☐Quiz ☐Presentation ☐Project ☐Published work
Introduction / Theoretical basics	1. About the first problem, students have to consider that the volume of the cuboid will be 125 *l* =125 dm^3 and that the volume can be calculated with the formula *V=abc* where *a*, *b* and *c* are the dimensions of the cuboid. A sketch of the cuboid is necessary. They also have to consider that the amount of material needed for the tin tank construction is equal to the tank's area. The area can be calculated with the formula *P=2(ab+bc+ac)*. Thus, students have to minimize the area of the rectangular cuboid with fixed volume. Students are encouraged to use Excel and formulas in it to consider how the area is changing with the change of cuboide's dimensions, while the volume is fixed. One example for such Excel spreadsheet is following:	

a	b	c	V	P
2	3	20,83333	125	220,3333333
3	4	10,41667	125	169,8333333
4	5	6,25	125	152,5
5	6	4,166667	125	151,6666667
6	7	2,97619	125	161,3809524
7	8	2,232143	125	178,9642857
8	9	1,736111	125	203,0277778
9	10	1,388889	125	232,7777778

10	9	1,388889	125	232,7777778
7	7	2,55102	125	169,4285714
8	8	1,953125	125	190,5
6	6	3,472222	125	155,3333333
5	**5**	**5**	**125**	**150**
4	4	7,8125	125	157
10	12	1,041667	125	285,8333333
9	6	2,314815	125	177,4444444
5	4	6,25	125	152,5
4	4	7,8125	125	157
6	9	2,314815	125	177,4444444

Students can create different charts with the data in the table, using Excel:



According to the values in the table, students can realize that the area is the smallest when three dimensions of the rectangular cubic are equal. But, this conclusion needs scientific support…

2. About the second problem, students' have to consider that the area of the box is 200 cm^2 and have to remind that we can calculate the area of the box with the formula *P=2(ab+bc+ac)* where *a*, *b* and *c* are the dimensions of the box. Students have to

determine the dimensions of the box which gives the largest volume. The volume is calculated with the formula $V=abc$. Thus, students have to maximize the volume.

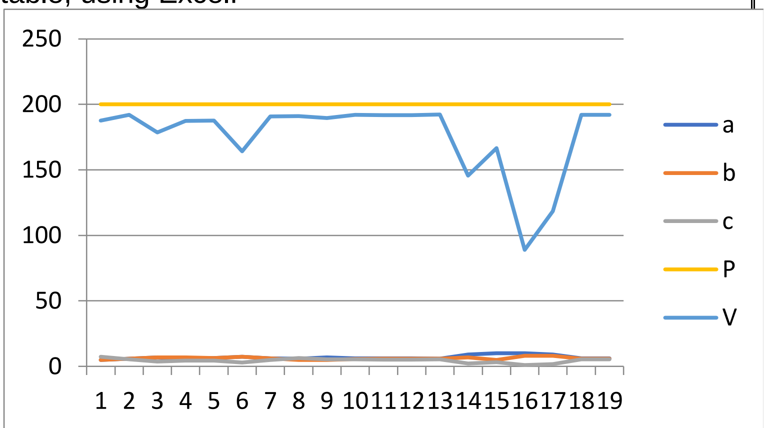
As well as in the first problem, students are encouraged to use digital tools in order to determine the solution easier. They can use Excel, too, for considering how the volume is changing with the change of the dimensions, while the area is fixed.

One example for such Excel spreadsheet is following:

a	***b***	***c***	***P***	***V***
5	5	7,5	200	187,5
6	6	5,333333	200	192
7	7	3,642857	200	178,5
6	7	4,461538	200	187,3846154
6,5	6,5	4,442308	200	187,6875
7,5	7,5	2,916667	200	164,0625
6,2	6,2	4,964516	200	190,836
6	5	6,363636	200	190,9090909
7	5	5,416667	200	189,5833333
6,2	5,7	5,433613	200	192,0238992
6,2	5,9	5,241322	200	191,7275702
6,1	6	5,239669	200	191,7719008
6	**5,9**	**5,428571**	**200**	**192,1714286**
9	7	2,3125	200	145,6875
10	5	3,333333	200	166,6666667
10	8	1,11111	200	88,88888889

		1		
9	8	1,647059	200	118,5882353
6,1	5,9	5,334167	200	191,9766583
6,1	5,8	5,430252	200	192,1223193

Students can create different charts with the data in the table, using Excel:



It is a little bit difficult for giving conclusion about the result in this problem, although it will be heuristically done.

Thus, mathematical support, algorithms and formulas are necessary for exact calculation of the solution in both of the problems.

Such minimizing and maximizing problems can easily be solved with an application of derivatives of functions with several variables. In these certain problems, we apply functions with two variables.

If $z = f(x, y)$ is given function with two independent variables, we determine its minimum/maximum values with the next algorithm: 1) we first calculate partial derivatives of first order, i.e. $z'_x = \frac{\partial z}{\partial x}$ and $z'_y = \frac{\partial z}{\partial y}$; 2)

	we solve the system of equations $\frac{\partial z}{\partial x}=0$ and $\frac{\partial z}{\partial y}=0$. The solutions of the system give us possible extreme values and are called stationary points; 3) we calculate the partial derivatives of the second order for the function $z=f(x,y)$, i.e. $z''_{xx}=\frac{\partial^2 z}{\partial x^2}$, $z''_{xy}=\frac{\partial^2 z}{\partial y \partial x}$, $z''_{yy}=\frac{\partial^2 z}{\partial y^2}$ and $z''_{yx}=\frac{\partial^2 z}{\partial x \partial y}$; 4) we calculate the value of each partial derivative of the second order in each stationary point. $z''_{xy}=z''_{yx}$ is necessary condition for the function to reach minimum/maximum value in certain stationary point. 5) we calculate the determinant $\Delta=\begin{vmatrix} z''_{xx}(x_0,y_0) & z''_{xy}(x_0,y_0) \\ z''_{yx}(x_0,y_0) & z''_{yy}(x_0,y_0) \end{vmatrix}$ for each stationary point (x_0,y_0); 6) if $\Delta<0$ in the considered stationary point, the function doesn't reach an extreme value in that point. If $\Delta=0$ we cannot conclude anything about the extreme value in that point and if $\Delta>0$ we conclude that the function has extreme value in the considered stationary point; 7) in order to be $\Delta>0$, because $z''_{xy}(x_0,y_0)=z''_{yx}(x_0,y_0)$, thus the sign of $z''_{xx}(x_0,y_0)$ and $z''_{yy}(x_0,y_0)$ must be the same. If $z''_{xx}(x_0,y_0)>0$ then the function reaches local minimum in the considered stationary point. If $z''_{xx}(x_0,y_0)<0$ then the function reaches local maximum in the considered stationary point. If we consider certain size as function with to variables, we can calculate its extreme values with previously described procedure.	
Action	For the first given problem, we know that $V=abc$ and $V=125$, thus $abc=125$, i.e. $c=\frac{125}{ab}$. We have to minimize the area, and the area is $P=2(ab+bc+ac)$. Substituting $c=\frac{125}{ab}$, we have	

<table>
<tr><td></td><td>$P = 2\left(ab + b\cdot\frac{125}{ab} + a\cdot\frac{125}{ab}\right)$, i.e. $P = 2\left(ab + \frac{125}{a} + \frac{125}{b}\right)$. Now we consider the area as a function with two real variables and following the algorithm above, we calculate that for $a = b = c = 5$ we have minimal area.

In a same way we can solve the second problem.</td><td rowspan="2"></td></tr>
<tr><td>Materials / equipment / digital tools / software</td><td>Literature given in the references at the end of the document /
Digital device which supports Excel /
Excel</td></tr>
<tr><td>Consolidation</td><td colspan="2">With the given examples students can consider that the real functions and their derivatives are important for solving real life problems. Students will learn how to calculate partial derivatives and how to apply differentiation and derivatives to maximize / minimize certain value by given conditions. Students can use technology, different digital tools and software as a help for solving problems, but can also realize that even with technology, solving different everyday problems is difficult without math knowledge.</td></tr>
<tr><td colspan="3">Reflections and next steps</td></tr>
<tr><td colspan="2">Activities that worked</td><td>Parts to be revisited</td></tr>
<tr><td colspan="2">Problem solving, collaboration, using technology</td><td>Depends on the students, in a conversation with students the teacher will realize the difficulties that students had and then revisit appropriate parts.</td></tr>
<tr><td colspan="3">References</td></tr>
<tr><td colspan="3">[1] E. Atanasova, S. Georgieva (2002), Matematika 2, Universitet “Sv. Kiril I Metodij” - Skopje
[2] S. Calaway D. Hoffman and D.Lippman (2014) “Applied Calculus”
[3] P.D. Lax, M. S.Terrell (2014) “Calculus with Applications”, Springer</td></tr>
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